Chapter Nine

Trigonometric Ratio

In our day to day life we make use of triangles, and in particular, right angled triangles. Many different examples from our surroundings can be drawn where right triangles can be imagined to be formed. In ancient times, with the help of geometry men learnt the technique of determining the width of a river by standing on its bank. Without climbing the tree they knew how to measure the height of the tree accuratly by comparing its shadow with that of a stick. In all the situations given above, the disttances or heights can be found by using some mathematical technique which come under a special branch of mathematics called Trigonometry. The word 'Trigonometry' is derived from Greek words 'tri' (means three), 'gon' (means edge) and `metron' (means measure). In fact, trigonometry is the study of relationship between the sides and angles of a triangle. There are evidence of using the Trigonometry in Egyptian and Babilian civilization. It is believed that the Egyptians made its extensive use in land survey and engineering works. Early astrologer used it to determine the distances from the Earth to the farest planets and stars. At present trigonometry is in use in all branches of mathematics. There are wide usages of trigonometry for the solution of triangle related problems and in navigation etc. Now a days trigonometry is in wide use in Astronomy and Calculus.

At the end of the chapter, the students will be able to –

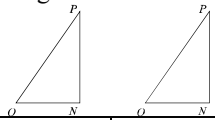
- Describe the trigonometric ratios of acute angles
- Determine the mutual relations among the trigonometric ratios of acute angle
- Solve and prove the mathematical problems justifying the trigonometric ratios of acute angle
- Determine and apply trigonometric ratios of acute angles 30^o. 45^o, 60^o geometrically
- Determine and apply the value of meaningful trigonometric ratios of the angles 0^o and 90^o
- Prove the trigonometric identities
- Apply the trigonometric identities.

9·1 Naming of sides of a right angled triangle

We know that, the sides of right angles triangle are known as hypotenuse, base and height. This is successful for the horizontal position of triangle. Again, the naming of sides is based on the position of one of the two acute angles of right angled triangle. As for example :

a. 'Hypotenuse', the side of a right angled triangle, which is the opposite side of the right angle.

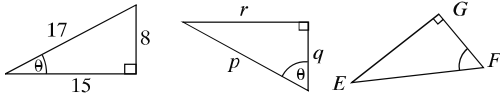
b. 'Opposite side', which is the direct opposite side of a given angle.

c. 'Adjacent side', which is a line segment constituting the given angle.

For the angle $\angle PON$, OP is the hypotenuse, ON is the adjacent side and PN is the opposite side.	For the angle $\angle OPN$, OP is the hypotenuse, PN is the adjacent side and ON is the opposite side.

In the geometric figure, the capital letters are used to indicate the vertices and small letters are used to indicate the sides of a triangle. We often use the Greek letters to indicate angle. Widely used six letters of Greek alphabet are :

alpha α	beta β	gamma γ	theta θ	phi ϕ	omega ω

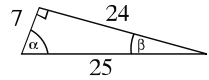
Greek letter are used in geometry and trigonometry through all the great mathematician of ancient Greek.

Example 1. Indicate the hypotenuse, the adjacent side and the opposite side for the angle θ.

Solution:

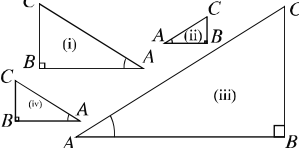
(a) hypotenuse 17 units
opposite side 8 units
adjacent side 15 units

(b) hypotenuse p
opposite side r
adjacent side q

(c) hypotenuse EF
opposite side EG
adjacent side FG

Example 2. Find the lengths of hypotenuse, the adjacent side and the opposite side for the angles α and β.

(a) For α angle,
hypotenuse 25 units
opposite side 24 units
adjacent side 7 units.

(b) For β angle
hypotenuse 25 units
opposite side 7 units
adjacent side 24 units.

Activity :
Indicate the hypotenuse , adjacent side and opposite for the angle θ and ϕ.

9·2 Constantness of ratios of the sides of similar right-angled triangles

Activity : Measure the lengths of the sides of the following four similar triangles and complete the table below. What do you observe about the ratios of the triangles ?

length of sides			ratio (related to angle)		
BC	AB	AC	BC/AC	AB/AC	BC/AB

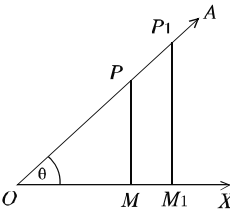
Let, $\angle XOA$ is an acute angle. We take a point P on the side OA. We draw a perpendicular from P to OX. As a result, a right angled triangle POM is formed. The three ratios of the sides PM, OM and OP of ΔPOM do not depend on the position of the point P on the side OA.

If we draw the perpendiculars PM and P_1M_1 from two points P and P_1 of OA to the side OX, two similar triangles ΔPOM and ΔP_1OM_1 are formed.

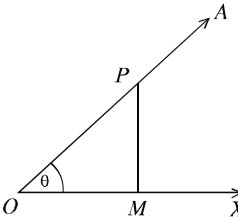
Now, ΔPOM and ΔP_1OM_1 are being similar,

$\frac{PM}{P_1M_1} = \frac{OP}{OP_1}$ or, $\frac{PM}{OP} = \frac{P_1M_1}{OP_1}$ ..... (i)

$\frac{OM}{OM_1} = \frac{OP}{OP_1}$ or, $\frac{OM}{OP} = \frac{OM_1}{OP_1}$ .... (ii)

$\frac{PM}{P_1M_1} = \frac{OM}{OM_1}$ or, $\frac{PM}{OM} = \frac{P_1M_1}{OM_1}$ .... (iii)

That is, each of these ratios is constant. These ratios are called trigonometric ratios.

9·3 Trigonometric ratios of an acute angle

Let, $\angle XOA$ is an acute angle. We take any point P on OA. We draw a perpendicular PM from the point P to OX. So, a right angled triangle POM is formed. The six ratios are obtained from the sides PM, OM and OP of ΔPOM which are called trigonometric ratios of the angle $\angle XOA$ and each of them are named particularly.

With respect to the $\angle XOA$ of right angled triangle POM, PM is the opposite side. OM is the adjacent side and OP is the hypotenuse. Denoting $\angle XOA = \theta$, the obtained six ratios are described below for the angle

From the figure :

$$\sin\theta = \frac{PM}{OP} = \frac{\text{opposite side}}{\text{hypotenuse}} \quad [\text{sine of angle } \theta]$$

$$\cos\theta = \frac{OM}{OP} = \frac{\text{adjacent side}}{\text{hypotenuse}} \quad [\cos\textit{ine} \text{ of angle } \theta]$$

$$\tan\theta = \frac{PM}{OM} = \frac{\text{opposite side}}{\text{adjacent side}} \quad [\tan\textit{gent} \text{ of angle } \theta]$$

And opposite ratios of them are

$$\text{cosec}\theta = \frac{1}{\sin\theta} \quad [\text{cosec}\textit{ant} \text{ of angle } \theta]$$

$$\sec\theta = \frac{1}{\cos\theta} \quad [\sec\textit{ant} \text{ of angle } \theta]$$

$$\cot\theta = \frac{1}{\tan\theta} \quad [\cot\textit{angent} \text{ of angle } \theta]$$

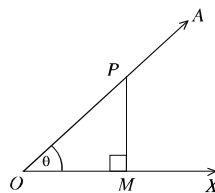
We observe, the symbol $\sin\theta$ means the ratio of sine of the angle θ, not the multiplication of sin and θ. sin is meaningless without θ. It is applicable for the other trigonometric ratios as well.

9·4 Relation among the trigonometric ratios

Let, $\angle XOA = \theta$ is an acute angle.
from the adjacent figure, according to the definition

$$\sin\theta = \frac{PM}{OP}, \quad \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{OP}{PM}$$

$$\cos\theta = \frac{OM}{OP}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{OP}{OM}$$

$$\tan\theta = \frac{PM}{OM}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{OM}{PM}$$

Again, $\tan\theta = \frac{PM}{OM} = \frac{\frac{PM}{OP}}{\frac{OM}{OP}}$ [Dividing the numerator and the denominator by OP]

$$= \frac{\sin\theta}{\cos\theta}$$

$$\boxed{\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}}$$

and similarly

$$\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$$

9·5 Trigonometric identity

(i) $(\sin\theta)^2+(\cos\theta)^2 = \left(\frac{PM}{OP}\right)^2+\left(\frac{OM}{OP}\right)^2$

$= \frac{PM^2}{OP^2}+\frac{OM^2}{OP^2} = \frac{PM^2+OM^2}{OP^2} = \frac{OP^2}{OP^2}$ [by the formula of Pythagoras]

$=1$

or, $(\sin\theta)^2+(\cos\theta)^2=1$

$$\therefore\ \sin^2\theta+\cos^2\theta=1$$

Remark : For integer indices n we can write $\sin^n\theta$ for $(\sin\theta)^n$ and $\cos^n\theta$ for $(\cos\theta)^n$.

(ii) $\sec^2\theta=(\sec\theta)^2 = \left(\frac{OP}{OM}\right)^2$

$= \frac{OP^2}{OM^2} = \frac{PM^2+OM^2}{OM^2}$ [OP is the hypotenuse of right angled ΔPOM]

$= \frac{PM^2}{OM^2}+\frac{OM^2}{OM^2}$

$= 1+\left(\frac{PM}{OM}\right)^2 = 1+(\tan\theta)^2 = 1+\tan^2\theta$

$\therefore$ $\sec^2\theta=1+\tan^2\theta$

or, $\sec^2\theta-\tan^2\theta=1$

or, $\tan^2\theta=\sec^2\theta-1$

(iii) $\text{cosec}^2\theta = (\text{cosec}\,\theta)^2 = \left(\frac{OP}{PM}\right)^2$

$= \frac{OP^2}{PM^2} = \frac{PM^2+OM^2}{PM^2}$ [is the hypotenuse of right angled ΔPOM]

$= \frac{PM^2}{PM^2}+\frac{OM^2}{PM^2} = 1+\left(\frac{OM}{PM}\right)^2$

$= 1+(\cot\theta)^2 = 1+\cot^2\theta$

$\therefore$ $\text{cosec}^2\theta-\cot^2\theta=1$ and $\cot^2\theta=\text{cosec}^2\theta-1$

Activity :
1. Construct a table of the following trigonometric formulae for easy memorizing.

$\text{cosec}\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	$\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ $\text{cosec}^2\theta = 1 + \cot^2\theta$

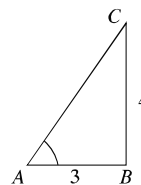
Example 1. If $\tan A = \dfrac{4}{3}$, find the other trigonometric ratios of the angle A.

Solution : Given that, $\tan A = \dfrac{4}{3}$.

So, opposite side of the angle $A = 4$, adjacent side $= 3$

hypotenuse $= \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Therefore,, $\sin A = \dfrac{4}{5}$, $\cos A = \dfrac{3}{5}$, $\cot A = \dfrac{3}{4}$

$\text{cosec}\, A = \dfrac{5}{4}$, $\sec A = \dfrac{5}{3}$.

Example 2. $\angle B$ is the right angle of a right angled triangle ABC. If $\tan A = \dfrac{3}{4}$, verify the truth of $2\sin A \cos A = 1$.

Solution : Given that, $\tan A = \dfrac{3}{4}$,

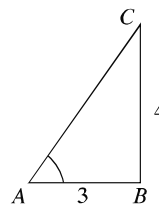
So, opposite side of the angle $A = 4$, adjacent side $= 3$

hypotenuse $= \sqrt{4^2 + 3^2} = \sqrt{5} = 5$

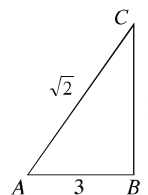
so, $\sin A = \dfrac{4}{5}$, $\cos A = \dfrac{3}{4}$

Hence, $2 \sin A \cos A = 2 \cdot \dfrac{4}{5} \cdot \dfrac{3}{5} = \dfrac{24}{25} \neq 1$

Therefore, $2\sin A \cos A = 1$ is a false statement.

Example 3. $\angle B$ is a right angle of a right angled triangle ABC. If $\tan A = 1$, verify the justification of $2\sin A \cos A = 1$.

Solution : Given that, $\tan A = 1$.
So, opposite side of the $A = 1$, adjacent side $= 1$
hypotenuse $= \sqrt{1^2 + 1^2} = \sqrt{2}$
Therefore, $\sin A = \frac{1}{\sqrt{2}}$, $\cos A = \frac{1}{\sqrt{2}}$.

Now left hand side $= 2\sin A \cos A = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 2 \cdot \frac{1}{2} = 1$

= right hand side.
$\therefore$ $2\sin A\cos A = 1$ is a true statement.

Activity :
1. If $\angle C$ is a right angle of a right angled triangle ABC, $AB = 29$ cm, $BC = 21$ cm and $\angle ABC = \theta$, find the value of $\cos^2\theta - \sin^2\theta$.

Example 4. Prove that, $\tan\theta + \cot\theta = \sec\theta \cdot \operatorname{cosec}\theta$.

Proof :
Left hand side $= \tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cdot \cos\theta}$$
$$= \frac{1}{\sin\theta \cdot \cos\theta} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$
$$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$$
$$= \operatorname{cosec}\theta \cdot \sec\theta$$
$$= \sec\theta \cdot \operatorname{cosec}\theta =$$

= R.H.S. (proved)

Example 5. Prove that, $\frac{1}{1+\sin^2\theta} + \frac{1}{1+\operatorname{cosec}^2\theta} = 1$

Proof : L.H.S. $= \frac{1}{1+\sin^2\theta} + \frac{1}{1+\operatorname{cosec}^2\theta}$

$$= \frac{1}{1+\sin^2\theta} + \frac{1}{1+\frac{1}{\sin^2\theta}}$$
$$= \frac{1}{1+\sin^2\theta} + \frac{\sin^2\theta}{1+\sin^2\theta}$$

$$= \frac{1+\sin^2\theta}{1+\sin^2\theta}$$

$= 1 =$ R.H.S. (proved)

Example 6. Prove that : $\dfrac{1}{2-sin^2 A} + \dfrac{1}{2+tan^2 A} = 1$

Proof : L.H.S. $= \dfrac{1}{2-sin^2 A} + \dfrac{1}{2+tan^2 A}$

$$= \frac{1}{2-sin^2 A} + \frac{1}{2+\dfrac{sin^2 A}{cos^2 A}}$$

$$= \frac{1}{2-sin^2 A} + \frac{cos^2 A}{2cos^2 A + sin^2 A}$$

$$= \frac{1}{2-sin^2 A} + \frac{cos^2 A}{2(1-sin^2 A) + sin^2 A}$$

$$= \frac{1}{2-sin^2 A} + \frac{cos^2 A}{2-2sin^2 A + sin^2 A}$$

$$= \frac{1}{2-sin^2 A} + \frac{1-sin^2 A}{2-sin^2 A}$$

$$= \frac{2-sin^2 A}{2-\sin^2 A}$$

$= 1 =$ R.H.S. (proved)

Example 7. Prove that : $\dfrac{tanA}{secA+1} - \dfrac{secA-1}{tanA} = 0$

proof : L.H.S. $= \dfrac{tanA}{secA+1} - \dfrac{secA-1}{tanA}$

$$= \frac{tan^2 A - (sec^2 A - 1)}{(secA+1)tanA} \quad [\sec^2\theta - 1 = \tan^2\theta]$$

$$= \frac{tan^2 A - tan^2 A}{(secA+1)tanA}$$

$$= \frac{0}{(secA+1)tanA}$$

$= 0 =$ R.H.S. (proved)

Example 8. Prove that : $\sqrt{\dfrac{1-sinA}{1+sinA}} = secA - tanA$

Prove : L.H.S. $= \sqrt{\dfrac{1-sinA}{1+sinA}}$

$= \sqrt{\dfrac{(1-sinA)(1-sinA)}{(1+sinA)(1-sinA)}}$ [Multiplying the numerator and the denominator by $\sqrt{(1-sinA)}$]

$= \sqrt{\dfrac{(1-sinA)^2}{1-sin^2 A}}$

$= \sqrt{\dfrac{(1-sinA)^2}{cos^2 A}}$

$= \dfrac{1-\text{sinA}}{\text{cosA}}$

$= \dfrac{1}{cosA} - \dfrac{sinA}{cosA}$

$= secA - tanA$

= R.H.S. (proved).

Example 9. If $\text{tanA}+\text{sinA}=\text{a}$ and $tanA - sinA = b$, prove that, $a^2 - b^2 = 4\sqrt{ab}$.

Prove : We given that, $tanA + sinA = a$ and $tanA - sinA = b$

L.H.S. $= a^2 - b^2$

$= (tanA + sinA)^2 - (tanA - sinA)^2$

$= 4\text{tanA sinA}$ $[\because (a-b)^2 - (a+b)^2 = 4ab]$

$= 4\sqrt{tan^2 A\, sin^2 A}$

$= 4\sqrt{tan^2 A\,(1-cos^2 A)}$

$= 4\sqrt{tan^2 A - tan^2 A \cdot cos^2 A}$

$= 4\sqrt{tan^2 A - sin^2 A}$

$= 4\sqrt{(tanA + sinA)(tanA - sinA)}$

$= 4\sqrt{ab}$

= R.H.S. (proved)

Activity : 1. If $\cot^4 A - \cot^2 A = 1$, prove that, $\cos^4 \theta + \cos^2 A = 1$
2. If $\sin^2 A - \sin^4 A = 1$, prove that, $\tan^4 A + \tan^2 A = 1$

Example 10. If $secA + tanA = \dfrac{5}{2}$, find the value of $secA - tanA$.

Solution: We given that, $secA + tanA = \dfrac{5}{2}$.............$(i)$

We know that, $sec^2 A = 1 + tan^2 A$

or, $sec^2 A - tan^2 A = 1$
or, $(secA + tanA)(secA - tanA) = 1$
or, $\frac{5}{2}(secA - tanA) = 1$ [from (i)]

$\therefore \quad secA - tanA = \frac{2}{5}$

Exercise 9·1

1. Verify whether each of the following mathematical statements is true or false. Give argument in favour of your answer.
   (a) The value of $\tan A$ is always less than 1.
   (b) $\cot A$ is the multiplication of cot and A.
   (c) For any value of A, $\sec A = \frac{12}{5}$.
   (d) cos is the smallest form of cotangent.
2. If $\sin A = \frac{3}{4}$, find the other trigonometric ratios of the angle A.
3. Given that $15 \cot A = 8$, find the values of $\sin A$ and $\sec A$.
4. If $\angle C$ is the right angle of the right angled triangle ABC, $AB = 13$ cm and $BC = 12$ cm. and $\angle ABC = \theta$, find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.
5. $\angle B$ is the right angle of the right angled triangle ABC. If $\tan A = \sqrt{3}$, verify the truth of $\sqrt{3} \sin A \cos A = 4$.

Prove (6 – 20) :

6. (i) $\frac{1}{\sec^2 A} + \frac{1}{\text{cosec}^2 A} = 1$; (ii) $\frac{1}{\cos^2 A} - \frac{1}{\cot^2 A} = 1$; (iii) $\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} = 1$;
7. (i) $\frac{\sin A}{\text{cosec} A} + \frac{\cos A}{\sec A} = 1$; (ii) $\frac{\sec A}{\cos A} - \frac{\tan A}{\cot A} = 1$. (iii) $\frac{1}{1+\sin^2 A} + \frac{1}{1+\text{cosec}^2 A} = 1$
8. (i) $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \sec A \cdot \text{cosec} A + 1$; (ii) $\frac{1}{1+\tan^2 A} + \frac{1}{1+\cot^2 A} = 1$
9. $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$.
10. $\tan A \sqrt{1 - \sin^2 A} = \sin A$.
11. $\frac{\sec A + \tan A}{\text{cosec} A + \cot A} = \frac{\text{cosec} A - \cot A}{\sec A - \tan A}$
12. $\frac{\text{cosec} A}{\text{cosec} A - 1} + \frac{\text{cosec} A}{\text{cosec} A + 1} = 2\sec^2 A$.
13. $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$.
14. $\frac{1}{\text{cosec} A - 1} - \frac{1}{\text{cosec} A + 1} = 2\tan^2 A$.
15. $\frac{\sin A}{1-\cos A} + \frac{1-\cos A}{\sin A} = 2 \text{ cosec } A$.
16. $\frac{\tan A}{\sec A + 1} - \frac{\sec A - 1}{\tan A} = 0$

17. $(\tan\theta + \sec\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$ 18. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \, . \tan B.$

19. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$ 20. $\sqrt{\dfrac{\sec A + 1}{\sec A - 1}} = \cot A + \operatorname{cosec} A.$

21. If $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \sqrt{2}\sin A$

22. If $\tan A = \dfrac{1}{\sqrt{3}}$, find the value of $\dfrac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}$.

23. If $\operatorname{cosec} A - \cot A = \dfrac{4}{3}$, what is the value of $\operatorname{cosec} A + \cot A$?

24. If $\cot A = \dfrac{b}{a}$, find the value of $\dfrac{a\sin A - b\cos A}{a\sin A + b\cos A}$.

9·6 Trigonometric ratios of the angles 30°, 45° and 60°

We have learnt to draw the angles having the measurement of 30°, 45° and 60° geometrically. The actual values of the trigonometric ratios for all these angles can be determined geometrically.

Trigonometric ratios of the angles 30° and 60°

Let, $\angle XOZ = 30^\circ$ and P is a point on the side OZ.
Draw $PM \perp OX$ and extend PM upto Q such that $MQ = PM$. Add O, Q and extend upto Z.
Now, between ΔPOM and ΔQOM, $PM = QM$, OM is the common side and included $\angle PMO$ = included $\angle QMO = 90^\circ$

$\therefore\ \Delta POM \cong \Delta QOM$

Therefore, $\angle QOM = \angle POM = 30^\circ$

and $\angle OQM = \angle OPM = 60^\circ$

Again, $\angle POQ = \angle POM + \angle QOM = 30^\circ + 30^\circ = 60^\circ$

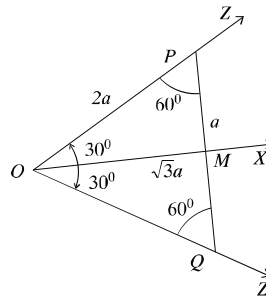
$\therefore\ \Delta OPQ$ is an equilateral triangle.

If $OP = 2a$, $PM = \dfrac{1}{2}PQ = \dfrac{1}{2}OP = a$ [since ΔPOQ is an equilateral triangle]

From right-angled ΔOPM, we get,

$OM = \sqrt{OP^2 - PM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$.

We find the trigonometric ratios :

$\therefore\ \sin 30^\circ = \dfrac{PM}{OP} = \dfrac{a}{2a} = \dfrac{1}{2}$, $\cos 30^\circ = \dfrac{OM}{OP} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$

$$\tan 30^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}.$$

$$\text{cosec } 30^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2, \ \sec 30^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}.$$

Similarly,

$$\sin 60^\circ = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{PM}{OP} = \frac{a}{2a} = \frac{1}{2}, \tan 60^\circ = \frac{OM}{PM} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\text{cosec } 60^\circ = \frac{OP}{OM} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}},$$

$$\sec 60^\circ = \frac{OP}{PM} = \frac{2a}{a} = 2, \cot 60^\circ = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}.$$

Trigonometric ratio of the angle 45°

Let, $\angle XOZ = 45^\circ$ and P is a point on OZ.
Draw $PM \perp OX$. In right angled triangle ΔOPM, $\angle POM = 45^\circ$

So, $\angle OPM = 45^\circ$

Therefore, $PM = OM = a$ (suppose)

Now, $OP^2 = OM^2 + PM^2 = a^2 + a^2 = 2a^2$

or, $OP = \sqrt{2}\, a$

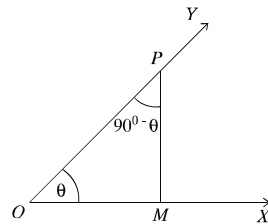
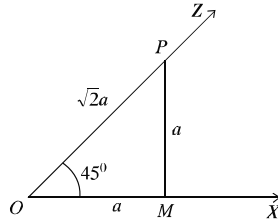
From the definition of trigonometric ratios, we get

$$\sin 45^\circ = \frac{PM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \ \cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}, \ \tan 45^\circ = \frac{PM}{OM} = \frac{a}{a} = 1$$

$$\text{cosec } 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}, \ \sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}, \ \cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

9·7 Trigonometric ratios of complementary angles

We know, if the sum of two acute angles is 90°, one of them is called complementary angle to the other. For example, 30° and 60°; 15° and 75° are complementary angles to each other.

In general, the angles θ and $(90^\circ - \theta)$ are complementary angles to each other.

Trigonometric ratios of complementary angles

Let, $\angle XOY = \theta$ and P is the point on the side OY of the angle. We draw $PM \perp OX$.

Since the sum of the three angles of a triangle is two right angles therefore, in the right angled triangle POM, $\angle PMO = 90^\circ$

and $\angle OPM + \angle POM =$ one right angle $= 90^\circ$

$\therefore \ \angle OPM = 90^\circ - \angle POM = 90^\circ - \theta$

[Since $\angle POM = \angle XOY = \theta$]

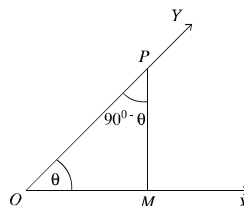
$\therefore \quad \sin\ (90^\circ - \theta) = \dfrac{OM}{OP} = \cos \angle POM = \cos\theta$

$\cos (90^\circ - \theta) = \dfrac{PM}{OP} = \sin \angle POM = \sin\theta$

$\tan\ (90^\circ - \theta) = \dfrac{OM}{PM} = \cot \angle POM = \cot\ \theta$

$\cot\ (90^\circ - \theta) = \dfrac{PM}{OM} = \tan \angle POM = \tan\ \theta$

$\sec (90^\circ - \theta) = \dfrac{OP}{PM} = \operatorname{cosec} \angle POM = \operatorname{cosec}\ \theta$

$\operatorname{cosec} (90^\circ - \theta) = \dfrac{OP}{OM} = \sec \angle POM = \sec\ \theta$.

We can express the above formulae in words below :

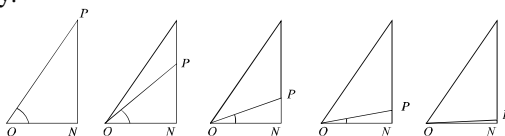
sine of complementary angle = *cosine* of angle

cosine of complementary angle = *sine* of angle

tangent of complementary angle = *cotangent* of angle etc.

Activity : 1. If $\sec (90^\circ - \theta) = \dfrac{5}{3}$**,** find the value of $\operatorname{cosec}\theta - \cot\theta$ **.**

9·8 Trigonometric ratios of the angles 0° and 90°

We have learnt how to determinal the trigonometric ratios for the acute angle θ of a right angled triangle. Now, we see, if the angle is made gradually smaller, how the trigonometric ratios change. As θ get smallers the length of the side PN also gets smaller. The point P closes to the point N and finally the angle θ comes closer to the angle 0°, OP is reconciled with ON approximately.

When the angle θ comes closer to 0°, the length of the line segment PN reduces to zero and in this case the value of $\sin\theta \frac{PN}{OP}$ is approximately zero. At the same time, the length of OP is equal to the length of ON and the value of $\cos\theta = \frac{ON}{OP}$ is 1 approximately.

The angle, 0° is introduced for the convenience of discussion in trigonometry, and the edge line and the original line of the angle 0° are supposed the same ray. Therefore, in line with the prior discussion, it is said that, $\cos 0^\circ = 1$, $\sin 0^\circ = 0$.

If θ is the acute angle, we see

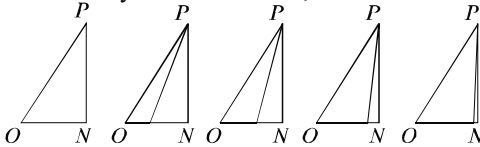
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta},$$

We define the angle 0° in probable cases so that, those relations exists.

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1.$$

Since division by 0 is not allowed, cosec 0° and $\cot 0^\circ$ can not be defined.

Again, when the angle θ is very closed to 90°, hypotenuse OP is approximatel y equal to PN. So the value of $\sin\theta$ is approximately 1. On the other hand, if the angle θ is is equal to 90°, ON is nearly zero;the value of $\cos\theta$ is approximately 0. So, in agreement of formulae that are described above, we can say, $\cos 90^\circ = 0$, $\sin 90^\circ = 1$.

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

$$\operatorname{cosec} 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1$$

Since one can not divided by 0, as before, $\tan 90^\circ$ and $\sec 90^\circ$ are not defined.

Observe : For convenience of using the values of trigonometric ratios of the angles 0°, 30°, 45°, 60° and 90° are shown in the following table :

angle / Ratio	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cosecant	undefined	$\frac{1}{2}$	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Observe : Easy method for remembering of the values of trigonometric ratios of some fixed angles.

(i) If we divide the numbers $0, 1, 2, 3$ and 4 by 4 and take square root of the quotients, we get the values of $\sin 0^\circ$ of $\sin 30^\circ$, $\sin 45^\circ$, $\sin 60^\circ$ and $\sin 90^\circ$ respectively.

(ii) If we divide the numbers $4, 3, 2, 1$ and 0 by 4 and take square root of quotients, we get the values of $\cos 0^\circ, \cos 30^\circ$, $\cos 45^\circ$, $\cos 60^\circ$ and $\cos 90^\circ$ respectively.

(iii) If we divide the numbers $0, 1, 3$ and 9 by 3 and take square root of quotients, we get the values of $\tan 0^\circ$, $\tan 30^\circ$, $\tan 45^\circ$ and $\tan 60^\circ$, respectively (It is noted that tan 90° is undefined).

(iv) If we divide the numbers $9, 3, 1$ and 0 by 3 and take square root of quotients, we get the values of $\cot 30^\circ$, $\cot 45^\circ$, $\cot 60^\circ$, $\cot 90^\circ$ respectively (It is noted that $\cot 0^\circ$ is undefined).

Example 1. Find the values :

(a) $\dfrac{1 - sin^2 45^\circ}{1 + sin^2 45^\circ} + tan^2 45^\circ$

(b) $cot\, 90^\circ \cdot tan\, 0^\circ \cdot sec\, 30^\circ \cdot cosec\, 60^\circ$

(c) $sin\, 60^\circ \cdot cos\, 30^\circ + cos\, 60^\circ \cdot sin\, 30^\circ$

(d) $\dfrac{1 - tan^2 60^\circ}{1 + sin^2 60^\circ} + sin^2 60^\circ$

Solution :

(a) Given expression = $\dfrac{1-sin^2 45^\circ}{1+sin^2 45^\circ}+tan^2 45^\circ$

$$=\frac{1-\left(\frac{1}{\sqrt{2}}\right)^2}{1+\left(\frac{1}{\sqrt{2}}\right)^2}+(1)^2 \quad [\because sin45^\circ=\frac{1}{\sqrt{2}} \text{ I } tan45^\circ=1]$$

$$=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}+1=\frac{\frac{1}{2}}{\frac{3}{2}}+1=\frac{1}{3}+1=\frac{4}{3}$$

(b) Given expression = $cot\,90^\circ\cdot tan\,0^\circ\cdot sec\,30^\circ\cdot cosec\,60^\circ$

$$=0\cdot 0\cdot\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}=0$$

$$[\because cot\,90^\circ=0,\ tan\,0^\circ=0,\ sec\,30^\circ=\frac{2}{\sqrt{3}},\ cosec\,60^\circ=\frac{2}{\sqrt{3}}]$$

(c) Given expression = $sin\,60^\circ\cdot cos\,30^\circ+cos\,60^\circ\cdot sin\,30^\circ$

$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot\frac{1}{2}$$

$$[\because \sin 60^\circ=\cos 30^\circ=\frac{\sqrt{3}}{2},\ \sin 30^\circ=\cos 60^\circ=\frac{1}{2}]$$

$$=\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1$$

(d) Given expression = $\dfrac{1-\tan^2 60^\circ}{1+\sin^2 60^\circ}+\sin^2 60^\circ$

$$=\frac{1-(\sqrt{3})^2}{1+(\sqrt{3})^2}+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$=\frac{1-3}{1+3}+\frac{3}{4}=\frac{-2}{4}+\frac{3}{4}$$

$$=\frac{-2+3}{4}=\frac{1}{4}$$

Example 2.

(a) If $\sqrt{2}\cos(\mathrm{A}-\mathrm{B})=1,\ 2\sin(\mathrm{A}+\mathrm{B})=\sqrt{3}$ and A, B are acute angles, find the values of A and B.

(b) If $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$, find the value of A.

(c) Prove that, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$, if $A = 45^\circ$.

(d) Solve : $2\cos^2\theta + 3\sin\theta - 3 = 0$, where θ is an acute angle.

Solution : (a) $\sqrt{2}\cos(A-B) = 1$

or, $\cos(A-B) = \frac{1}{\sqrt{2}}$

or, $\cos(A-B) = \cos 45^\circ$ $[\because \cos 45^\circ = \frac{1}{\sqrt{2}}]$

$\therefore$ $A - B = 45^\circ$..................(i)

and $2\sin(A+B) = \sqrt{3}$

or, $\sin(A+B) = \frac{\sqrt{3}}{2}$

or, $\sin(A+B) = \sin 60^\circ$ $[\because \sin 60^\circ = \frac{\sqrt{3}}{2}]$

$\therefore$ $A + B = 60^\circ$....................(ii)

Adding (i) and (ii), we get,

$2A = 105^\circ$

$\therefore$ $A = \frac{105^\circ}{2} = 52\frac{1}{2}^\circ$

Again, subtracting (i) from (ii), we get

$2B = 15^\circ$

or, $B = \frac{15^\circ}{2}$

$\therefore$ $B = 7\frac{1}{2}^\circ$

Required $A = 52\frac{1}{2}^\circ$ and $B = 7\frac{1}{2}^\circ$

(b) $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$

or, $\frac{\cos A - \sin A + \cos A - \sin A}{\cos A - \sin A - \cos A - \sin A} = \frac{1-\sqrt{3}+1-\sqrt{3}}{1-\sqrt{3}+1-\sqrt{3}}$

or, $\dfrac{2\cos A}{-2\sin A}=\dfrac{2}{-2\sqrt{3}}$

or, $\dfrac{\cos A}{\sin A}=\dfrac{1}{\sqrt{3}}$

or, $\cot A=\cot 60^\circ$

$\therefore$ $A=60^\circ$

(c) Given that, $A=45^\circ$

we have to prove that, $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

L.H.S. = $\cos 2A$

$=\cos(2\times 45^\circ)=\cos 90^\circ=0$

R.H.S. = $\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$=\dfrac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ}=\dfrac{1-(1)^2}{1+(1)^2}$

$=\dfrac{0}{2}=0$

$\therefore$ L.H.S. = R.H.S. (proved)

(d) Given equation, $2\cos^2\theta+3\sin\theta-3=0$

or, $2(1-\sin^2\theta-3(1-\sin\theta)=0$

or, $2(1+\sin\theta)(1-\sin\theta)-3(1-\sin\theta)=0$

or, $(1-\sin\theta)\{2(1+\sin\theta)-3\}=0$

or, $(1-\sin\theta)\{2\sin\theta-1\}=0$

or, $1-\sin\theta=0$ or $2\sin\theta-1=1$

$\therefore$ $\sin\theta=1$ or, $2\sin\theta=1$

or, $\sin\theta=\sin 90^\circ$ or, $\sin\theta=\dfrac{1}{2}$

$\therefore$ $\theta=90^\circ$ or, $\sin\theta=\sin 30^\circ$

or, $\theta=30^\circ$

θ is an acute angle, so $\theta=30^\circ$.

Exercise 9·2

1. If $\cot\theta=\dfrac{1}{2}$, which one is the value of $\cot\theta$?

(a) $\dfrac{1}{\sqrt{3}}$ (b) 1 (c) $\sqrt{3}$ (d) 2

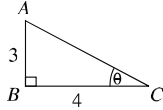
2.

(i) $\sin^2\theta = 1 - \cos^2\theta$

(ii) $\sec^2\theta = 1 + \tan^2\theta$

(iii) $\cot^2\theta = 1 - \tan^2\theta$

Which one of the followings is correct in accordance with the above statements.

(a) i and ii (b) i and iii (c) ii and iii (d) i, ii and iii

Answer of questions 3 and 4 on the basis of the figure :

3. What is the value of $\sin\theta$?

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

4. What is the value of $\cot\theta$?

(a) $\frac{3}{4}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{4}{3}$

Evaluate (5 – 8) :

5. $\frac{1-\cot^2 60^\circ}{1-\cot^2 60^\circ}$.

6. $\tan 45^\circ \cdot \sin^2 60^\circ \cdot \tan 30^\circ \cdot \tan 60^\circ$.

7. $\frac{1-\cos^2 60^\circ}{1-\cos^2 60^\circ} + \sec^2 60^\circ$.

8. $\cos 45^\circ \cdot \cot^2 60^\circ \cdot \operatorname{cosec}^2 30^\circ$.

Prove (9 – 11) :

9. $\cos^2 30^\circ - \sin^2 30^\circ = \cos 60^\circ$.

10. $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ = \sin 90^\circ$.

11. $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ = \cos 30^\circ$.

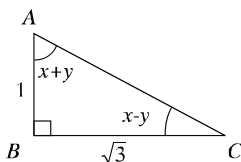
12. $\sin 3A = \cos 3A$, if $A = 15^\circ$.

13. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, if $A = 45^\circ$.

14. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$, if $A = 30^\circ$.

15. $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$, if $A = 60^\circ$.

16. If $2\cos(A+B) = 1 = 2\sin(A-B)$ and A, B are acute angles, show that $A = 45^\circ$, $B = 15^\circ$.

17. If $\cos(A-B)=1$, $2\sin(A+B)$ and A, B are acute angle, find the values of A and B.
18. Solve : $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.
19. If A, B are acute angle and $\cot(A+B)=1$, $\cot(A-B)=\sqrt{3}$, find the values of A and B.
20. Show that, $\cos 3A=4\cos^3 A-3\cos A$, when $A=30^\circ$.
21. Solve : $\sin\theta+\cos\theta=1$, when $0^\circ\leq\theta\leq 90^\circ$.
22. Solve : $\cos^2\theta-\sin^2\theta=2-5\cos\theta$, when θ is an acute angle.
23. Solve : $2\sin^2\theta+3\cos\theta-3=0$, θ is an acute angle.
24. Solve: $\tan^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=0$.
25. Find the value : $3\cot^2 60^\circ+\frac{1}{4}\operatorname{cosec}^2 30^\circ+5\sin^2 45^\circ-4\cos^2 60^\circ$.
26. If $\angle B=90^\circ$, $AB=5\text{cm}$, $BC=12\text{cm}$. of ΔABC
    (a) Find the length of AC.
    (b) If $\angle C=\theta$, find the value of $\sin\theta+\cos\theta$.
    (c) Show that, $\sec^2\theta+\cos^2\theta=\sec^2\theta\cdot\operatorname{cosec}^2\theta$.
27.

(a) what is the measurement of AC.
(b) Find the value of $\tan A+\tan C$.
(c) Find the values of x and y.